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Effects of differential rotation on the eigenfrequencies of small adiabatic barotropic modes of oscillations of polytropic models of stars

A K Lal[1], **Ankush Pathania**[1], **Alka Bhalla**[2] **and C Mohan**[3,4]

[1] School of Mathematics and Computer Applications, Thapar University, Patiala, Punjab, India
[2] Department of Mathematics, N.I.T., Jalandhar, Punjab, India
[3] Professor of Mathematics (Retd.), IIT Roorkee, Uttarakhand, India

E-mail: aklal@thapar.edu

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Abstract

Mohan *et al* (1992 *Astrophys. Space. Sci.* **193** 69) (1998 *Indian J. Pure Appl. Math.* **29** 199) investigated the problem of equilibrium structures and periods of small adiabatic oscillations of differentially rotating stellar models using a law of differential rotation of the type $\omega^2 = b_0 + b_1 s^2 + b_2 s^4$ (here ω is a nondimensional measure of the angular velocity of rotation of a fluid element at a distance s from the axis of rotation and $b's$ are suitably chosen constant parameters). This law of differential rotation assumes cylindrical symmetry for the rotating fluid elements. In the present paper, we have extended their study and used a more general law of differential rotation of the type $\omega^2 = b_0 + b_1 s^2 + b_2 s^4 + b_3 z^2 + b_4 z^4 + b_5 z^2 s^2$ in which the angular velocity of rotation of a fluid element is assumed to depend both on its distance s from the axis of rotation and on its distance z from the plane through the center of the star perpendicular to the axis of rotation. The main objective of this study has been to investigate whether the dependence of angular velocity of rotation on the parameter z in addition to the parameter s substantially alters the behavior of the eigenfrequencies of small adiabatic barotropic modes of oscillations of differentially rotating stars or not.

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1. Introduction

Some of the observed variable stars are also known to be rotating stars (for example FK Comae Berenices and BY Draconis). Whereas most of the rotating stars are expected to have solid

[4] Present Address: Ambala College of Engineering & Applied Research, Devasthali, Ambala Cantt., Haryana, India.

body rotations (the angular velocity of rotation of each point of the star is the same), some of the rotating stars are expected to be rotating differentially. In such type of stars different parts of the star rotate about the axis of rotation with different angular velocities. Rotation of a star is expected to produce some effect on the eigenfrequencies of oscillations of a variable star. However, effects produced by differential rotation are expected to depend upon the law of differential rotation that is used to appropriately reflect the rotation of the star. It may thus be of some interest to understand the effects of different types of differential rotations on the eigenfrequencies of small oscillations of stars.

The mathematical problem of determining the eigenfrequencies of oscillations of a rotating star is quite complex. Approximate methods have, therefore, been often used in the literature to study such problems. Most of the authors such as Clement (1965), Kochar and Trehan (1974), Saio (1981), Mohan and Saxena (1985), Lee and Saio (1987), Chandrasekhar and Ferrari (1991), Soofi *et al* (1998), Dintrans and Rieutord (2000), Reese *et al* (2006) and Lovekin and Deupree (2008) have studied the oscillations of stars assuming the star to have solid body rotation and therefore, rotating uniformly. However, some authors such as Clement (1967), Ireland (1967), Vorontsov (1983), Woodard (1989), Dziembowski and Goode (1992), Urpin *et al* (1996), Mohan *et al* (1998), Karino and Eriguchi (2003) and Lovekin *et al* (2009) addressed themselves to the problems of differentially rotating stars also.

Mohan *et al* (1992, 1998) studied the problems of the equilibrium structures and eigenfrequencies of small adiabatic barotropic modes of oscillations of polytropic models of stars assuming a law of differential rotation of the type

$$\omega^2 = b_0 + b_1 s^2 + b_2 s^4 \tag{1}$$

where the square of the angular velocity of rotation (ω) of a fluid element is assumed to depend upon the distance s of the fluid element from the axis of rotation.

In another study, Mohan *et al* (1994) (hereafter, referred as paper 1) extended this law of differential rotation to a more generalized type of law of differential rotation of the form

$$\omega^2 = b_0 + b_1 s^2 + b_2 s^4 + b_3 z^2 + b_4 z^4 + b_5 z^2 s^2 \tag{2}$$

in order to study the effects of more generalized laws of differential rotation on the equilibrium structures of differentially rotating polytropic models of stars. In this law of differential rotation, angular velocity of rotation (ω) depends not only upon the distance s of a fluid element from the axis of rotation but also on the distance z of this element from the equatorial plane. This law generates more general types of angular velocities of rotation of fluid elements inside the star. (In fact whereas law (1) is a Taylor series expansion of $\omega^2 = f(s^2)$, law (2) is a Taylor series expansion of $\omega^2 = f(s^2, z^2)$ up to second-order terms.)

In the case of a gaseous sphere undergoing periodic oscillations, two types of modes of oscillations are expected to be generated. One of these is called radial modes of oscillations (in which the fluid elements oscillate in the radial direction only) and the other nonradial modes (in which fluid elements oscillate in arbitrary directions). These nonradial modes are further classified into pressure p-modes (also called acoustic waves, here the restoring force is pressure) and gravitational g-modes (also called gravity waves, here the restoring force is gravity). It is expected that in the case of rotating stars (in which angular velocity of rotation is not too large) these types of modes are still excited but their eigenvalues get influenced by rotation effects.

While studying the eigenfrequencies of pseudo-radial and nonradial modes of oscillations of rotating polytropic models of stars, Mohan *et al* (1998) observed that the rotation causes eigenfrequencies of oscillations of g- modes to decrease. This conclusion is contrary to the conclusion of certain other authors who using certain other techniques have concluded that rotation increases the values of eigenfrequencies of oscillations of g-modes (cf Clement

(1984)). It is possible that this discrepancy in the eigenfrequencies of g-modes of nonradial oscillations might be due to neglect of the parameter z in the angular velocity of rotation which had not been considered by the authors in their law of differential rotation (1). It was, therefore, felt that it would be of some interest to analyze the effect of the parameter z in the angular velocity of rotation on the periods of small adiabatic barotropic modes of oscillations of differentially rotating stars.

Keeping this in mind, in the present paper we have computed the eigenfrequencies of pseudo-radial and nonradial modes of oscillations of polytropic models of stars rotating differentially according to law (2) where in addition to the parameter s, the effect of the parameter z is also taken into account in the expression for angular velocity of rotation.

The paper is organized as follows: in section 2 an eigenvalue boundary problem determining the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of a differentially rotating polytropic model of a star has been formulated using the law of differential rotation (2). An eigenvalue boundary problem which can be used to compute the eigenfrequencies of small adiabatic nonradial modes of oscillations of differentially rotating polytropic models of stars has next been obtained in section 3. Numerical computations have then been performed in section 4 to determine the eigenfrequencies of pseudo-radial and nonradial modes of oscillations of certain differentially rotating polytropic models of stars of polytropic indices 1.5 and 3.0 assuming different types of differential rotations obtained by assigning different values to the parameters $b's$ in (2). The eigenfrequencies thus computed have been compared with the earlier results obtained by Mohan *et al* (1998) (hereafter, referred as paper II) who did not consider the dependence of differential rotation on the parameter z, as well as original nonrotating models. Numerical results obtained have then been analyzed in section 5 and conclusions have been drawn.

2. Eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of differentially rotating polytropic models

Following Mohan *et al* (1998), an eigenvalue problem determining the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of a differentially rotating polytropic model of a star rotating differentially according to the law (2) may be expressed as

$$H_1\frac{d^2\varsigma}{dr_0^2} + H_2\frac{d\varsigma}{dr_0} + (H_3\omega^{*2} - H_4)\varsigma = 0. \tag{3}$$

Here $\omega^{*2} = \frac{R^3 r_{os}^3\sigma^2}{GM_0}$ and the expressions for rest of the symbols H_1, H_2, H_3, H_4 are given in appendix B, whereas ω^{*2} is the nondimensional form of the actual eigenfrequency of oscillation σ and ς denotes a suitable average of the relative amplitudes of pulsation of the fluid elements on the equipotential surface ψ = constant. Also r_{0s} is the value of r_0 at the surface of the model, G the universal gravitational constant, M_0 the total mass of the star and R the radius of undistorted polytropic model (necessary details of equation (3) are given in appendix A for readers' reference).

Equation (3) determines the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of differentially rotating polytropic models of stars rotating differentially according to law (2) which takes into account the dependence on the parameter z besides the parameter s. This equation is of Sturm–Liouville type and has to be solved subject to the boundary conditions which require ς to be finite at points corresponding to the center ($r_0 = 0$) and the free surface ($r_0 = r_{0s}$).

3. Eigenfrequencies of small adiabatic nonradial modes of oscillations of differentially rotating polytropic models

Following Mohan *et al* (1998) again, the system of differential equations governing the nonradial modes of oscillations of differentially rotating polytropic models of stars which are rotating differentially according to law (2) may be expressed as

$$
\begin{aligned}
&\frac{d\zeta}{dx} + B_1\zeta + \left(B_2 + \frac{B_3}{\omega^{*2}}\right)\eta + \frac{1}{\omega^{*2}}B_3\phi = 0\\
&\frac{d\eta}{dx} + (E_1\omega^{*2} + E_2)\zeta + E_3\eta + E_4\phi + \frac{d\phi}{dx} = 0\\
&\frac{d^2\phi}{dx^2} + F_1\frac{d\phi}{dx} + F_2\zeta + F_3\eta + F_4\phi = 0
\end{aligned} \tag{4}
$$

where ω^{*2} is the same as defined in (3) and the detailed expressions for the parameters B_1, B_2, E_1, E_2, etc obtained in series form for polytropic models are given in appendix B (necessary details of equation (4) are given in appendix A for readers' reference).

The eigenvalue problem (4) determines the eigenfrequencies of nonradial modes of oscillations of a differentially rotating polytropic model of a star rotating differentially according to law (2). It has to be solved subject to the boundary conditions.

At the center $x = 0$

$$
\eta + \phi = \frac{2\omega^2}{3lr_{0s}^4}\left(\frac{\overline{\rho}}{\rho_c}\right)\zeta, \qquad \frac{d\phi}{dx} = 0 \tag{5}
$$

and at the surface $x = 1$

$$
\begin{aligned}
\eta\, r_{0s}^3\Bigg[&1 + 2b_0\, r_{0s}^3 + \left(\frac{16b_1}{15} + \frac{8b_3}{15}\right)r_{0s}^5 + \frac{24b_0^2}{5}r_{0s}^6 + \left(\frac{16b_2}{21} + \frac{2b_4}{3} + \frac{8b_5}{21}\right)r_{0s}^7\\
&+ \left(\frac{44b_0b_1}{7} + \frac{44b_0b_3}{21}\right)r_{0s}^8 + \left(\frac{1664b_0b_2}{315} + \frac{104b_0b_4}{35} + \frac{208b_0b_5}{105} + \frac{104b_1b_3}{105}\right.\\
&\left.+ \frac{208b_1^2}{105} + \frac{52b_3^2}{35}\right)r_{0s}^{10} + \left(\frac{2240b_1b_2}{693} + \frac{203b_1b_4}{150} + \frac{224b_1b_5}{231} + \frac{448b_2b_3}{693}\right.\\
&\left.+ \frac{4b_3b_4}{3} + \frac{22\,072b_3b_5}{3465}\right)r_{0s}^{12} + \cdots\Bigg] + \frac{2}{\xi_u^2}\frac{d\theta_\psi}{dx}\zeta = 0
\end{aligned} \tag{6a}
$$

and

$$
\begin{aligned}
\frac{d\theta_\psi}{dx} + \phi\Bigg\{&l + (l+1)\Bigg[1 + b_0r_{0s}^3 + \left(\frac{2b_1}{3} + \frac{b_3}{3}\right)r_{0s}^5 + \frac{11b_0^2}{5}r_{0s}^6 + \left(\frac{8b_2}{15} + \frac{7b_4}{15} + \frac{4b_5}{15}\right)r_{0s}^7\\
&+ \left(\frac{368b_0b_1}{105} + \frac{104b_0b_3}{105}\right)r_{0s}^8 + \left(\frac{208b_0b_2}{63} + \frac{34b_0b_4}{21} + \frac{8b_0b_5}{7}\right.\\
&\left.+ \frac{52b_1b_3}{105} + \frac{44b_1^2}{35} + \frac{113b_3^2}{105}\right)r_{0s}^{10} + \left(\frac{38\,464b_1b_2}{17325} + \frac{56b_1b_4}{75} + \frac{3424b_1b_5}{5775}\right.\\
&\left.+ \frac{4208b_2b_3}{17325} + \frac{68b_3b_4}{75} + \frac{944b_3b_5}{1925}\right)r_{0s}^{12} + \cdots\Bigg]\Bigg\} = 0.
\end{aligned} \tag{6b}
$$

In the above expressions, terms up to second order of smallness in b_0, b_1, b_2, b_3, b_4, b_5 and up to order r_{0s}^{12} in r_{0s} have been retained. Other parameters used in equations (4)–(6) have the same meanings as assigned to them elsewhere in this paper.

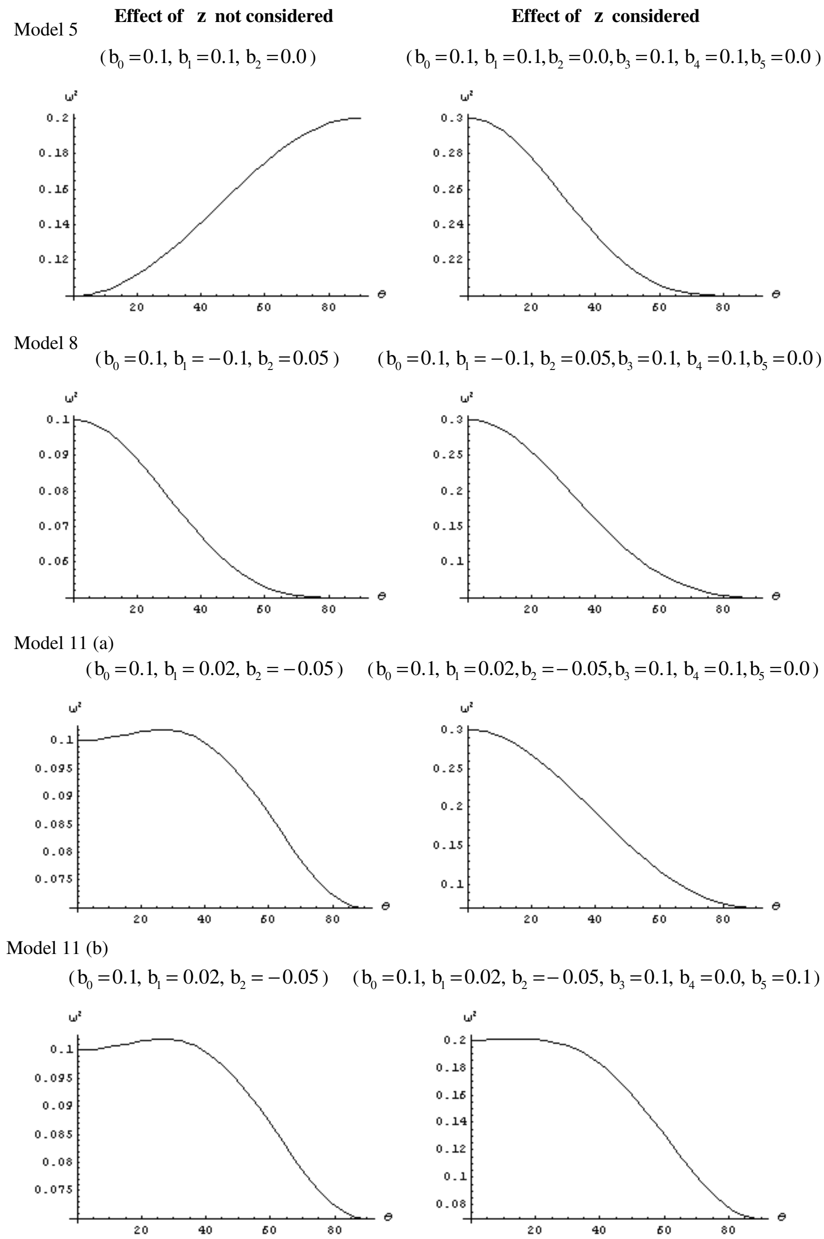

Figure 1. Variation of angular velocity ω^2 (in radians) with angle θ (in degrees).

On setting $b_3 = 0, b_4 = 0, b_5 = 0$ in (4–6), we get the same expressions as obtained earlier by Mohan *et al* (1998) (equation (8–12) of paper II) in which the dependence of angular velocity of rotation on the parameter z had not been considered.

The system of differential equations (4) along with the boundary conditions (5)–(6) may be used to compute the eigenfrequencies of nonradial modes of oscillations of differentially rotating polytropic models of stars rotating according to generalized law of differential rotation (2).

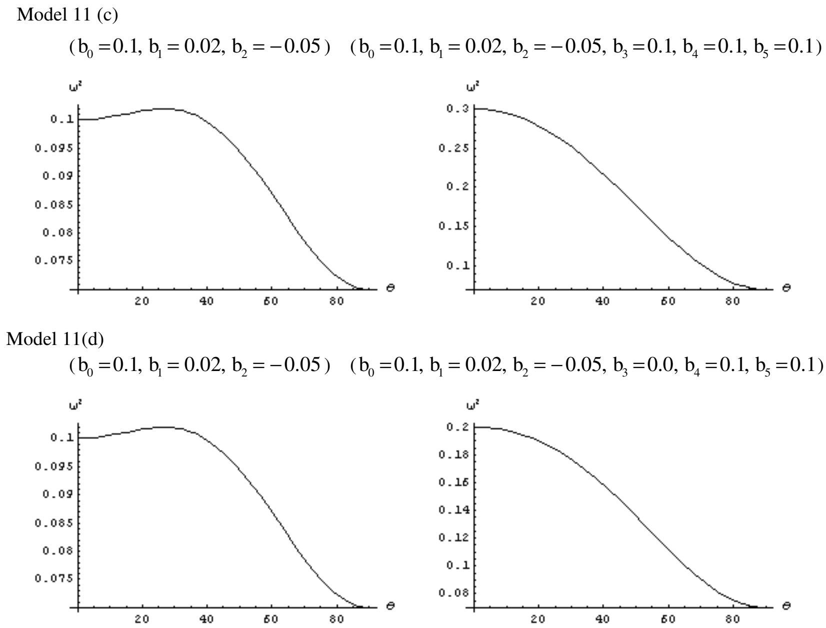

Figure 1. (Continued.)

4. Numerical evaluation of the eigenfrequencies

Eigenvalue problems developed in sections 2 and 3 have been solved numerically to compute eigenvalues of pseudo-radial and nonradial modes of oscillations of certain differentially rotating polytropic models. The eigenvalue problem of section 2 is of the Sturm–Liouville type. In order to compute the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of differentially rotating polytropic models, equation (3) has been integrated numerically subject to the boundary conditions which require ς to be finite at points corresponding to the center and the free surface of the model. The values of θ_ψ and $\mathrm{d}\theta_\psi/(\mathrm{d}x)$ needed for the purpose at various points were taken from the numerical solution of equation (19*a*) of paper I. Computations were started with some trial value of ω^2. For this chosen value of ω^2 a series solution was first developed at a point close to the center ($x = 0.005$). This solution is then used to carry the integration of the pulsation equation (3) outward using the fourth-order Runge–Kutta method. Using the same numerical value of ω^2, the series solution is also developed at points near the surface which is then used to carry the integration of equation (3) inward. The value of $\varsigma/(\mathrm{d}\varsigma/\mathrm{d}x)$ obtained from the outward and inward integrations of (3) is then matched at some preselected point in the interior of the model. The process is continued iteratively with different choices of the value of ω^2, till a value of ω^2 is found for which the two solutions agree to specified accuracy.

In order to start integrations from points near the center and the surface, series solutions were developed at $x = 0.01$ and $x = 0.99$. Outward and inward integrations were performed using a step length of $x = 0.01$. Trials with different values of ω^2 were continued till the absolute difference in the value of $\varsigma/(\mathrm{d}\varsigma/\mathrm{d}x)$ at the preselected point in the interior of the model from the outward and inward integrations was found to be less than 0.0005.

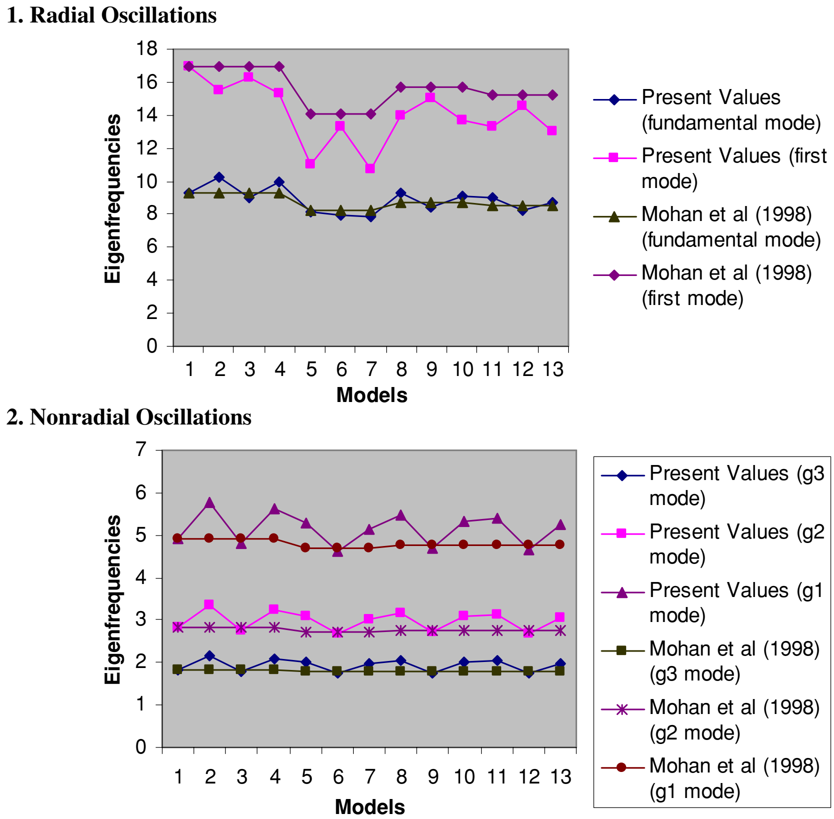

Figure 2. Comparison of the present values of the eigenfrequencies with the values of Mohan *et al* (1998) for polytropes of indices 3.0.

(This figure is in colour only in the electronic version)

Computations have been performed to compute the eigenfrequencies of the fundamental and the first mode of pseudo-radial modes of oscillations of differentially rotating polytropic models with polytropic indices 1.5 and 3.0 for different choices of rotation parameters $b's$ of law of differential rotation (2). (These values of $b's$ are listed in table 1.) The equilibrium structures of the differentially rotating polytropic models corresponding to these choices of the values of $b's$ have been earlier obtained in paper I. The obtained eigenfrequencies are presented in table 2. For comparison we also present in this table the corresponding results of Mohan *et al* (1998) who did not consider the effects of the parameter z in their law of differential rotation as well as corresponding nonrotating models.

Eigenfrequencies of the nonradial modes of oscillations have also been computed numerically using the eigenvalue problem of section 3. For this purpose the Chebyshev polynomial expansion technique used earlier by Mohan *et al* (1998) has been used. (The essential details of the method are given in Saxena (1984).) The boundary condition (6*a*) was used as the discriminant condition and $\zeta = 1$ at the center was used as the normalization condition. The values of θ_ψ and $(\mathrm{d}\theta_\psi/\mathrm{d}x)$ needed at various points in the interior of the model were taken from the solutions of the structure equation of these models obtained earlier in paper I. For polytropic models of indices 1.5 and 3.0 we normally used 10 and 15 collocation points, respectively. However, for determining the eigenfrequencies of certain higher modes

Table 1. Behavior of angular velocity in certain differentially rotating models.

	Values of various parameters in the law of differential rotation						Behavior of square of angular velocity of rotation ω^2 from center toward surface	
Model no	b_0	b_1	b_2	b_3	b_4	b_5	In equatorial plane ($z = 0, 0 \leqslant s \leqslant 1$) $\omega^2 = b_0 + b_1 s^2 + b_2 s^4$	Along axial direction ($s = 0, 0 \leqslant z \leqslant 1$) $\omega^2 = b_0 + b_3 z^2 + b_4 z^4$
1	0.0	0.0	0.0	0.0	0.0	0.0	No angular velocity	No angular velocity
2	0.0	0.0	0.0	0.1	0.1	0.0	No change	Increases from 0.0 at the center to 0.2 at the surface
3	0.0	0.0	0.0	0.1	0.0	0.1	No change	Increases from 0.0 at the center to 0.1 at the surface
4	0.0	0.0	0.0	0.1	0.1	0.1	No change	Increases from 0.0 at the center to 0.2 at the surface
5	0.1	0.1	0.0	0.1	0.1	0.0	Increases from 0.1 at the center to 0.2 at the surface	Increases from 0.1 at the center to 0.3 at the surface
6	0.1	0.1	0.0	0.1	0.0	0.1	Increases from 0.1 at the center to 0.2 at the surface	Increases from 0.1 at the center to 0.2 at the surface
7	0.1	0.1	0.0	0.1	0.1	0.1	Increases from 0.1 at the center to 0.2 at the surface	Increases from 0.1 at the center to 0.3 at the surface

Table 1. (Continued.)

Model no	Values of various parameters in the law of differential rotation						Behavior of square of angular velocity of rotation ω^2 from center toward surface	
	b_0	b_1	b_2	b_3	b_4	b_5	In equatorial plane $(z = 0, 0 \leqslant s \leqslant 1)$ $\omega^2 = b_0 + b_1 s^2 + b_2 s^4$	Along axial direction $(s = 0, 0 \leqslant z \leqslant 1)$ $\omega^2 = b_0 + b_3 z^2 + b_4 z^4$
8	0.1	−0.1	0.05	0.1	0.1	0.0	Decreases from 0.1 at the center to 0.05 at the surface	Increases from 0.1 at the center to 0.3 at the surface
9	0.1	−0.1	0.05	0.1	0.0	0.1	Decreases from 0.1 at the center to 0.05 at the surface	Increases from 0.1 at the center to 0.2 at the surface
10	0.1	−0.1	0.05	0.1	0.1	0.1	Decreases from 0.1 at the center to 0.05 at the surface	Increases from 0.1 at the center to 0.3 at the surface
11	0.1	0.02	−0.05	0.1	0.1	0.0	Increases from 0.1 at the center to 0.102 at $s = 0.45$ and then decreases to 0.07 at the surface	Increases from 0.1 at the center to 0.3 at the surface
12	0.1	0.02	−0.05	0.1	0.0	0.1	Increases from 0.1 at the center to 0.102 at $s = 0.45$ and then decreases to 0.07 at the surface	Increases from 0.1 at the center to 0.2 at the surface
13	0.1	0.02	−0.05	0.1	0.1	0.1	Increases from 0.1 at the center to 0.102 at $s = 0.45$ and then decreases to 0.07 at the surface	Increases from 0.1 at the center to 0.3 at the surface

Model 1 is an undistorted model.

Table 2. Eigenfrequencies $\omega^{*2} = r_{os}^3 R^3 \sigma^2/(GM_0)$ for the fundamental mode $\omega_0^{*\,2}$ and the first mode $\omega_1^{*\,2}$ of pseudo-radial modes of oscillations of differentially rotating polytropic models of stars.

Model no	$N = 1.5$		$N = 3.0$	
	$\omega_0^{*\,2}$	$\omega_1^{*\,2}$	$\omega_0^{*\,2}$	$\omega_1^{*\,2}$
1	2.7059 (2.7059)	12.5338 (12.5338)	9.2550 (9.2550)	16.9885 (16.9885)
2	2.9320 (2.7059)	10.2591 (12.5338)	10.2455 (9.2550)	15.5011 (16.9885)
3	2.6206 (2.7059)	12.0455 (12.5338)	8.9724 (9.2550)	16.3226 (16.9885)
4	2.8512 (2.7059)	10.2463 (12.5338)	9.9394 (9.2550)	15.2718 (16.9885)
5	2.4594 (2.4046)	7.7887 (10.2210)	8.1257 (8.2244)	11.0217 (14.0875)
6	2.3271 (2.4046)	9.7016 (10.2210)	7.9218 (8.2244)	13.2873 (14.0875)
7	2.4048 (2.4046)	7.6674 (10.2210)	7.8819 (8.2244)	10.7504 (14.0875)
8	2.6755 (2.5438)	9.5215 (11.5492)	9.3130 (8.7137)	13.9663 (15.7109)
9	2.4664 (2.5438)	11.0509 (11.5492)	8.4411 (8.7137)	15.0356 (15.7109)
10	2.6152 (2.5438)	9.3915 (11.5492)	9.0681 (8.7137)	13.6958 (15.7109)
11	2.5972 (2.4906)	9.1197 (11.1341)	8.9885 (8.5301)	13.2991 (15.1941)
12	2.4134 (2.4906)	10.6355 (11.1341)	8.2494 (8.5301)	14.5091 (15.1941)
13	2.5400 (2.4906)	9.0024 (11.1341)	8.7486 (8.5301)	13.0251 (15.1941)

Results shown in parenthesis correspond to the values of eigenfrequencies earlier computed by Mohan *et al* (1998) assuming a differential rotation law of type (1) in which the parameter z had not been taken into account.

of nonradial oscillations, the number of collocation points was further increased to achieve the desired accuracy of 0.0001 in getting the discriminant condition satisfied. The number of collocation points used in determining a specific mode of nonradial oscillation of a distorted polytropic model was, however, kept the same as used in determining the corresponding mode for the corresponding undistorted model.

The results for the nonradial modes of oscillations of certain differentially rotating polytropic models of indices 1.5 and 3.0 are presented in tables 3 and 4 respectively. The corresponding results when the dependence of angular velocity of rotation on the parameter z was not considered (Mohan *et al* 1998) are also presented in these tables for comparison.

The variations in the angular velocities ω^2 (for different values of $b's$ as given in table 1) with angle θ (the angle which the point P (x, y, z) makes with the positive x-axis at the center of the differentially rotating star, here r is taken to be 1) are shown in figure 1 for some of the considered models. The effect of considering the parameter z in the law of differential rotation on the eigenfrequencies of pseudo-radial and nonradial modes is also depicted graphically in figure 2.

5. Analysis of the numerical results and conclusions

In this paper, we have computed the eigenfrequencies of various differentially rotating polytropic models of stars in which the angular velocity of rotation depends on two parameters s and z by assuming a general law of differential rotation of type (2). The eigenfrequencies of these differentially rotating polytropic models of indices 1.5 and 3.0 have been compared with the corresponding eigenfrequencies of an undistorted polytropic model (model 1) as well as differentially rotating models developed by Mohan *et al* (1998) using a law of differential rotation of type (1) so as to determine the effect of inclusion of the parameter z (distance of a fluid element from the equatorial plane) in the law of differential rotation on the eigenfrequencies of pseudo-radial and nonradial modes of oscillation of differentially rotating polytropic models of stars. (Polytropic models of indices 3.0 and 1.5 are generally considered to be reasonably good approximations of inner structures of massive and less massive stars on and near the main sequence.)

Our results in table 2 show that in the case of differentially rotating stars whose angular velocity of rotation ω depends on both the parameters z and s, the eigenfrequencies of pseudo-radial modes of oscillations of various differentially rotating polytropic models of stars considered by us decrease in comparison to the eigenfrequencies of an undistorted polytropic model. This conclusion is similar to the earlier conclusions of Mohan *et al* (1998) who only considered the dependence of differential rotation on the parameter s.

Results presented in tables 3 and 4 show that differential rotation, in general, decreases the values of eigenfrequencies of f- and p-modes of nonradial oscillations of polytropic models of indices 1.5 and 3.0 considered by us as compared to the corresponding values of undistorted polytropic models. This behavior is similar to the behavior reported earlier by Mohan *et al* (1998) in the case of differentially rotating stars whose angular velocity depends on the parameter s alone.

Results of table 4 also show that when the angular velocity of rotation depends more on the parameter z then there is an increase in the values of eigenfrequencies of g-modes of nonradial oscillations of differentially rotating polytropic models of stars as compared to the corresponding values of an undistorted polytropic model whereas there is a decrease in the values when the dependence of angular velocity of rotation on the parameter z is less.

Our results in table 4 thus show that with the inclusion of the parameter z in the law of differential rotation, the eigenfrequencies of g-modes of nonradial oscillations of differentially rotating polytropic models, in general, are greater than the corresponding values of an undistorted model. These results are contrary to the results earlier reported by Mohan *et al* (1998). However, these results are in accordance with the results of authors such as Clement (1984) who have reported that on account of rotation there is, in general, an increase in the values of g-modes of nonradial oscillations of stars.

On comparing our results in tables 2–4 with the corresponding results obtained earlier by Mohan *et al* (1998), we found that, in general, with the inclusion of the parameter z in the law of differential rotation the eigenfrequencies of fundamental and first modes of pseudo-radial oscillations and f- and p-modes of nonradial oscillations decrease. However, for polytropic models of index 3.0 with the inclusion of the parameter z, the eigenfrequencies of f-modes increase in certain models and decrease in others. Table 4 also shows that, in general, there is an increase in the values of eigenfrequencies of g-modes of nonradial oscillations with the inclusion of the parameter z as compared to the corresponding values earlier obtained by Mohan *et al* (1998). Some of the results of tables 3 and 4 for polytropes of index 3.0 have also been depicted in figure 2.

Our present study has thus shown that, in general, with the dependence of angular velocity of rotation on the parameter z in addition to the parameter s, the eigenfrequencies of pseudo-radial and nonradial modes (f- and p-modes) of oscillations of differentially rotating polytropic models of indices 1.5 and 3.0, in general, decrease on account of rotation whereas eigenfrequencies of g-modes of nonradial oscillations increase on account of rotation.

Table 3. Eigenfrequencies $\omega^{*2} = r_{os}^3 R^3 \sigma^2/(GM_0)$ of nonradial modes of oscillations of differentially rotating polytropic models ($N = 1.5$).

Model no	f	p_1	p_2	p_3
1	2.4225 (0–0) (2.4225)	10.2911 (1–1) (10.2911)	23.5163 (2–2) (23.5163)	41.2956 (3–3) (41.2956)
2	2.0617 (2.4225)	7.1194 (10.2911)	14.9219 (4–2) (23.5163)	22.9184 (5–3) (41.2956)
3	2.2985 (2.4225)	9.7687 (10.2911)	22.3301 (23.5163)	38.5240 (41.2956)
4	2.0116 (2.4225)	7.0795 (10.2911)	14.7299 (4–2) (23.5163)	23.4966 (5–3) (41.2956)
5	1.6761 (1.9946)	5.9426 (8.2984)	12.8248 (18.9733)	39.3759 (5–3) (33.7827)
6	1.8972 (1.9946)	7.9130 (8.2984)	18.0511 (18.9733)	31.2721 (33.7827)
7	1.6447 (1.9946)	5.9233 (8.2984)	12.7422 (18.9733)	40.1572 (5–3) (33.7827)
8	1.8903 (2.2087)	6.8698 (9.4254)	14.3954 (4–2) (21.5167)	22.4400 (5–3) (36.6377)
9	2.0932 (2.2087)	8.9450 (9.4254)	20.3759 (21.5167)	34.2721 (36.6377)
10	1.8403 (2.2087)	6.7934 (9.4254)	14.3828 (4–2) (21.5167)	23.5307 (36.6377)
11	1.8402 (2.1504)	6.6370 (9.0807)	14.5736 (20.8506)	29.1000 (37.9741)
12	2.0376 (2.1504)	8.6195 (9.0807)	19.8009 (20.8506)	35.4345 (37.9741)
13	1.7937 (2.1504)	6.5658 (9.0807)	14.3525 (20.8506)	27.2000 (37.9741)

Numbers shown in parentheses are the number of nodes appearing in the eigenfunctions ζ and η. The case of the entries where no such nodes are shown indicates that these eigenfrequencies have the same number of nodes in ζ and η as are shown in the undistorted case (model 1).
Results shown in parenthesis correspond to the values of eigenfrequencies computed by Mohan *et al* (1998) assuming a differential rotation law of type (1) in which the parameter z had not been taken into account.

Table 4. Eigenfrequencies ω^{*2} of nonradial modes of oscillations of differentially rotating polytropic models ($N = 3.0$).

Model no	g_3	g_2	g_1	f	p_1	p_2	p_3
1	1.8358 (3–3) (1.8358)	2.8316 (2–2) (2.8316)	4.9264 (1–1) (4.9264)	8.2461 (0–0) (8.2461)	15.2827 (1–1) (15.2827)	26.7282 (2–2) (26.7282)	41.4744 (3–3) (41.4744)
2	2.1493 (1.8358)	3.3357 (2.8316)	5.7849 (4.9264)	8.8944 (4–0) (8.2461)	12.6491 (9–1) (15.2827)	18.8674 (10–4) (26.7282)	29.1967 (13–5) (41.4744)
3	1.7998 (1.8358)	2.7729 (2.8316)	4.8183 (4.9264)	7.9820 (8.2461)	14.5364 (15.2827)	25.3335 (4–2) (26.7282)	39.2490 (5–3) (41.4744)
4	2.0894 (1.8358)	3.2334 (2.8316)	5.6048 (4.9264)	8.6085 (4–0) (8.2461)	12.3222 (9–1) (15.2827)	18.8458 (12–4) (26.7282)	30.5049 (13–5) (41.4744)
5	2.0199 (1.7759)	3.1083 (2.7314)	5.2986 (4.7080)	7.2543 (8–4) (7.4062)	9.7420 (9–3) (12.4936)	15.3227 (12–4) (21.2997)	29.1590 (7–5) (32.8334) (5–3)
6	1.7439 (1.7759)	2.6789 (2.7314)	4.6061 (4.7080)	7.1460 (7.4062)	11.8018 (12.4936)	20.0237 (21.2997)	31.4239 (32.8334) (5–3)
7	1.9743 (1.7759)	3.0283 (2.7314)	5.1543 (4.7080)	7.0339 (8–6) (7.4062)	9.5222 (9–5) (12.4936)	15.9147 (10–8) (21.2997)	30.1528 (11–7) (32.8334) (5–3)
8	2.0472 (1.7908)	3.1656 (2.7570)	5.4628 (4.7800)	8.1745 (4–0) (7.7952)	11.5892 (9–1) (14.0400)	17.7551 (10–6) (24.3741) (4–2)	28.2581 (13–3) (37.5988) (9–3)

Table 4. (Continued.)

Model no	g_3	g_2	g_1	f	p_1	p_2	p_3
9	1.7597 (1.7908)	2.7069 (2.7570)	4.6820 (4.7800)	7.5457 (7.7952)	13.3163 (14.0400)	23.0027 (6–2) (24.3741) (4–2)	35.6656 (9–3) (37.5988) (9–3)
10	2.0000 (1.7908)	3.0880 (2.7570)	5.3238 (4.7800)	7.9382 (8–0) (7.7952)	11.4061 (9–1) (14.0400)	17.5552 (12–2) (24.3741) (4–2)	29.5655 (10–5) (37.5988) (9–3)
11	2.0336 (1.7839)	3.1448 (2.7462)	5.3964 (4.7508)	7.9024 (4–0) (7.6549)	11.0795 (9–1) (13.5410)	17.0187 (10– 2) (23.4990) (4–2)	27.5080 (13–3) (36.5066) (9–3)
12	1.7517 (1.7839)	2.6953 (2.7462)	4.6510 (4.7508)	7.4005 (7.6549)	12.8206 (13.5410)	22.1169 (23.4990) (4–2)	34.2086 (7–3) (36.5066) (9–3)
13	1.9889 (1.7839)	3.0661 (2.7462)	5.2566 (4.7508)	7.6633 (4–0) (7.6549)	10.7398 (9–1) (13.5410)	16.7665 (10–4) (23.4990) (4–2)	29.4162 (11–7) (36.5066) (9–3)

Numbers shown in parentheses are the number of nodes appearing in the eigenfunctions ζ and η. The case of the entries where no such nodes are shown indicates that these eigenfrequencies have the same number of nodes in ζ and η as are shown in the undistorted case (model 1).
Results shown in parenthesis correspond to the values of eigenfrequencies computed by Mohan *et al* (1998) assuming a differential rotation law of type (1) in which the parameter z had not been taken into account.

Appendix A. Eigenvalue boundary problems for computing pseudo-radial and nonradial modes of oscillations of differentially rotating models

Mohan *et al* (1991) formulated eigenvalue problems which determine the eigenfrequencies of small adiabatic pseudo-radial and nonradial modes of oscillations of rotationally and tidally distorted stellar models. The approach was later used by Mohan *et al* (1998) to determine the eigenfrequencies of small adiabatic pseudo-radial and nonradial modes of oscillations of certain differentially rotating stars. In this section we present in brief the approach adopted by Mohan *et al* (1991, 1998) to determine the eigenfrequencies of small adiabatic barotropic modes of oscillations of differentially rotating and tidally distorted stars.

A.1. Eigenvalue boundary problem determining the eigenfrequencies of small adiabatic barotropic pseudo-radial modes of oscillations of differentially rotating stars

Assuming that during oscillations the fluid elements on an equipotential surface oscillate in unison and keeping in view the fact that in hydrostatic equilibrium the equipotential surfaces are also surfaces of equipressure and equidensity (so that the values of pressure P and density ρ on the topologically equivalent spherical surfaces of the differentially rotating star are the actual values of P_ψ and ρ_ψ on its corresponding equipotential surfaces), the eigenfrequencies of small adiabatic pseudo-radial modes of oscillations of a differentially rotating stellar model can be obtained from its topologically equivalent spherical model developed on the basis of the averaging technique of Kippenhahn and Thomas (1970). Based on this approach, the equation determining the eigenfrequencies of pseudo-radial modes of oscillations of a differentially rotating stellar model is the same as the equation determining eigenfrequencies of radial modes of oscillations of the corresponding topologically equivalent spherical model and can be expressed as

$$\frac{\mathrm{d}^2\kappa}{\mathrm{d}r_{0\psi}^2} + \frac{4-\mu}{r_{0\psi}}\frac{\mathrm{d}\kappa}{\mathrm{d}r_{0\psi}^2} + \left[\frac{\rho_{0\psi}}{\gamma\, P_{0\psi}}\sigma^2 - \left(3-\frac{4}{\gamma}\right)\frac{\mu}{r_{0\psi}^2}\right]\kappa = 0 \tag{A.1}$$

where

$$\mu = -\frac{r_{0\psi}}{P_{0\psi}}\frac{\mathrm{d}P_{0\psi}}{\mathrm{d}r_{0\psi}}.$$

Here $r_{0\psi}$, $\rho_{0\psi}$ and $P_{0\psi}$ are the values of r_ψ, ρ_ψ and P_ψ on the equipotential surface $\psi =$ constant in its equilibrium position, γ is the ratio of specific heats, κ is the relative amplitude of pulsation of a fluid element distant r_ψ from the center of the topologically equivalent spherical model (it is thus some average of the relative amplitudes of pulsation of the fluid elements on the equipotential surface $\psi =$ constant of the actual distorted model) and σ is the eigenfrequency of an adiabatic barotropic pseudo-radial mode of small oscillation. The distance variable r_ψ of an element of the equivalent spherical model from its center is connected with the original parameters of the rotationally and tidally distorted model through the relation

$$\begin{aligned} r_\psi(b_0, b_1, b_2, r_0) = Rr_0\bigg[1 &+ \frac{b_0}{3}r_0^3 + \frac{2b_1}{15}r_0^5 + \frac{19b_0^2}{45}r_0^6 + \frac{8b_2}{105}r_0^7 + \frac{152b_0b_1}{315}r_0^8 \\ &+ \left(\frac{112b_0b_2}{315} + \frac{212b_1^2}{1575}\right)r_0^{10} + \frac{10\,144b_1b_2}{51\,975}r_0^{12} + \cdots\bigg] \end{aligned} \tag{A.2}$$

where $r_0 = \frac{1}{\psi - q}$,ψ being the nondimensional form of the total potential at the corresponding point of the distorted model, R is the undistorted equilibrium radius of the star of mass M_0 and $q = M_1/M_0$ is a nondimensional parameter representing the ratio of the mass of the secondary component over primary component of the binary system (we assume $q \ll 1$).

Using r_ψ, ρ_ψ and P_ψ in place of $r_{0\psi}$, $\rho_{0\psi}$ and $P_{0\psi}$ to denote the equilibrium values on the equipotential surfaces and taking r_0 in place of r_ψ, as the independent variable, equation (A.1) governing the small adiabatic pseudo-radial modes of oscillations of a differentially rotating star has been expressed by Mohan *et al* (1998) as

$$A(b_0,b_1,b_2,r_0)\frac{\mathrm{d}^2\kappa}{\mathrm{d}r_0^2}+\left[\frac{4-\mu}{r_0}B(b_0,b_1,b_2,r_0)-C(b_0,b_1,b_2,r_0)\right]\frac{\mathrm{d}\kappa}{\mathrm{d}r_0}$$
$$+\left[\frac{R^2\sigma^2\rho_\psi}{\gamma P_\psi}-\left(3-\frac{4}{\gamma}\right)\frac{\mu}{r_0^2}E(b_0,b_1,b_2,r_0)\right]\kappa=0 \tag{A.3}$$

where

$$A(b_0,\,b_1,\,b_2,\,r_0)=1-\frac{8b_0}{3}r_0^3-\frac{8b_1}{5}r_0^5-\frac{26b_0^2}{45}r_0^6-\frac{128b_2}{105}r_0^7-\frac{16\,b_0b_1}{7}r_0^8$$
$$-\left(\frac{928b_0b_2}{315}+\frac{328b_1^2}{315}\right)r_0^{10}-\left(\frac{22\,336\,b_1b_2}{10\,395}\right)r_0^{12}+\cdots$$

$$B(b_0,\,b_1,\,b_2,\,r_0)=1-\frac{5b_0}{3}r_0^3-\frac{14b_1}{15}r_0^5-\frac{47b_0^2}{45}r_0^6-\frac{24b_2}{35}r_0^7-\frac{136b_0b_1}{63}r_0^8$$
$$-\left(\frac{16b_0b_2}{7}+\frac{268b_1^2}{915}\right)r_0^{10}-\frac{16\,576b_1b_2}{10\,395}r_0^{12}-\cdots$$

$$C(b_0,\,b_1,\,b_2,\,r_0)=\frac{1}{r_0}\left[4b_0\,r_0^3+4b_1r_0^5+\frac{26b_0^2}{15}r_0^6+\frac{64b_2}{15}r_0^7+\frac{64b_0b_1}{7}r_0^8\right.$$
$$\left.+\left(\frac{928b_0b_2}{63}+\frac{328b_1^2}{63}\right)r_0^{10}+\frac{44\,672b_1b_2}{3465}r_0^{12}+\cdots.\right]$$

$$E(b_0,\,b_1,\,b_2,\,r_0)=1-\frac{2b_0}{3}r_0^3-\frac{4b_1}{15}r_0^5-\frac{23b_0^2}{45}r_0^6-\frac{16b_2}{105}r_0^7-\frac{44b_0b_1}{36}r_0^8$$
$$-\left(\frac{176b_0b_2}{315}+\frac{68b_1^2}{915}\right)r_0^{10}-\frac{3424b_1b_2}{10\,395}r_0^{12}-\cdots.$$

Also,

$$\mu=-\frac{r_\psi}{P_\psi}\frac{\mathrm{d}P_\psi}{\mathrm{d}r_0}\frac{\mathrm{d}r_0}{\mathrm{d}r_\psi}=-F\,(b_0,\,b_1,\,b_2)\frac{r_0}{P_\psi}\frac{\mathrm{d}P_\psi}{\mathrm{d}r_0}$$

where

$$F(b_0,\,b_1,\,b_2,\,r_0)=1-b_0r_0^3-\frac{2b_1}{3}r_0^5-\frac{6b_0^2}{5}r_0^6-\frac{56b_2}{105}r_0^7-\frac{76b_0b_1}{35}r_0^8$$
$$-\left(\frac{704b_0b_2}{315}+\frac{256b_1^2}{315}\right)r_0^{10}-\left(\frac{26\,144b_1b_2}{17\,325}\right)r_0^{12}-\cdots.$$

Equation (A.3) forms an eigenvalue boundary problem in the eigenfrequency of oscillation σ. This eigenvalue problem is of the Sturm–Liouville type having singularities both at the center and at the surface of the model. It has to be solved numerically subject to the boundary conditions which require κ to be finite at the center as well as at the free surface.

In reality equation (A.3) determines the periods of small adiabatic pseudo-radial modes of oscillations of the topologically equivalent spherical model. However, since equipotential surfaces of the actual rotationally and tidally distorted stellar model are also the surfaces of equipressure and equidensity, the values of pressure and density on the equipotential surfaces of the rotationally and tidally distorted stars are the same as on the corresponding equipotential surfaces of the equivalent spherical model. Hence, the eigenfrequencies of the pseudo-radial modes of oscillations determined by solving the eigenvalue problem for the topologically equivalent spherical model are indeed the eigenfrequencies of the pseudo-radial modes of oscillation of the undistorted model which has got distorted by the rotational and tidal effects. However, the values of the eigenfunction κ obtained on solving equation (A.3) are not the actual values of amplitudes of pulsation κ for the distorted model but rather some average of the true values of eigenfunction κ at various points on the corresponding equipotential surface of the rotationally and tidally distorted model.

We may thus use equation (A.3) to determine the effects of differential rotation on the periods of small adiabatic pseudo-radial modes of oscillations of a stellar model. The effects of differential rotation have been incorporated through introduction of terms $A(b_0, b_1, b_2, r_0)$, $B(b_0, b_1, b_2, r_0)$, $C(b_0, b_1, b_2, r_0)$, $E(b_0, b_1, b_2, r_0)$ and $F(b_0, b_1, b_2, r_0)$ as well as their influence on the values of ρ_ψ and P_ψ on ψ. The present method, in fact, incorporates the effects of distortional forces both while computing the equilibrium structure (in computing the values of P_ψ, ρ_ψ etc) and in solving (A.3) which determines the eigenfrequencies of oscillations.

The eigenvalue problem (A.3) together with the boundary conditions which require κ to be finite both at the center and at the free surface of the star can be solved numerically in the usual manner as is done in the case of undistorted models (for details see Lal (1993)).

A.2. Eigenvalue boundary problem determining the eigenfrequencies of small adiabatic barotropic nonradial modes of oscillations of differentially rotating stars

Mohan *et al* (1991) have also formulated the eigenvalued boundary value problem to determine the eigenfrequencies of nonradial modes of oscillations of rotationally and tidally distorted stellar models. As in the radial case, the values of the physical parameters ρ_ψ and P_ψ on the equipotential surfaces of the distorted model are assumed to be the same as those on the corresponding equipotential surfaces of the topologically equivalent spherical model. This topologically equivalent spherical model has then been used to determine the eigenfrequencies of nonradial modes of oscillations of the differentially rotating stellar models in the usual way. Mohan *et al* (1998) have expressed the eigenvalue problem determining the eigenfrequencies of nonradial modes of oscillations of differentially rotating stellar model in an explicit form as

$$\left.\begin{aligned}
&\frac{\mathrm{d}\zeta}{\mathrm{d}x} + B_1\zeta + \left(B_2 + \frac{1}{\sigma^2}B_3\right)\eta + \frac{1}{\sigma^2}B_3\phi = 0\\
&\frac{\mathrm{d}\eta}{\mathrm{d}x} + (E_1\sigma^2 + E_2)\zeta + E_3\eta + E_4\phi + \frac{\mathrm{d}\phi}{\mathrm{d}x} = 0\\
&\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + F_1\frac{\mathrm{d}\phi}{\mathrm{d}x} + F_2\zeta + F_3\eta + F_4\phi = 0
\end{aligned}\right\} \tag{A.4}$$

where

$$B_1 = \frac{l+1}{x} + \frac{1}{\gamma P_\psi}\frac{\mathrm{d}P_\psi}{\mathrm{d}x}$$

$$\begin{aligned}
B_2 &= \frac{2\pi G\rho_c}{Rx}\frac{\rho_\psi}{\gamma P_\psi} r_\psi^2 \frac{\mathrm{d}r_\psi}{\mathrm{d}x}\\
&= \frac{2\pi G\rho_c}{\gamma P_\psi} R^2 \rho_\psi r_{0s}^3 x \left[1 + 2b_0(xr_{0s})^3 + \frac{16b_1}{15}(xr_{0s})^5 + \frac{24b_0^2}{5}(xr_{0s})^6 + \frac{16b_2}{21}(xr_{0s})^7\right.\\
&\quad \left. + \frac{44\,b_0b_1}{7}(xr_{0s})^8 + \left(\frac{1664b_0b_2}{315} + \frac{208\,b_1^2}{105}\right)(xr_{0s})^{10} + \left(\frac{2240b_1b_2}{693}\right)(xr_{0s})^{12} + \cdots\right]
\end{aligned}$$

$$B_3 = -\frac{l\,(l+1)}{R\,x}\frac{\mathrm{d}r_\psi}{\mathrm{d}x}2\pi G\rho_c$$

$$= -\frac{l(l+1)}{x}2\pi\, G\, \rho_c r_{0s}\left[1+\frac{4b_0}{3}(xr_{0s})^3+\frac{4b_1}{5}(xr_{0s})^5+\frac{133b_0^2}{45}(xr_{0s})^6+\frac{64b_2}{105}(xr_{0s})^7\right.$$
$$\left.+\frac{152b_0b_1}{35}(xr_{0s})^8+\left(\frac{1232b_0b_2}{315}+\frac{2332b_1^2}{1575}\right)(xr_{0s})^{10}+\frac{131\,872b_1b_2}{51\,975}(xr_{0s})^{12}+\cdots\right]$$

$$E_1 = -\frac{1}{2\pi\, G\, \rho_c}\frac{R\, x}{r_\psi^2}\frac{\mathrm{d}r_\psi}{\mathrm{d}x}$$
$$= -\frac{1}{2\pi\, G\, \rho_c\, r_{0\,s}\, x}\left[1+\frac{2b_0}{3}(xr_{0s})^3+\frac{8b_1}{15}(xr_{0s})^5+\frac{14b_0^2}{9}(xr_{0s})^6+\frac{16b_2}{35}(xr_{0s})^7\right.$$
$$\left.+\frac{124b_0b_1}{45}(xr_{0s})^8+\left(\frac{96b_0b_2}{35}+\frac{184b_1^2}{175}\right)(xr_{0s})^{10}+\frac{9664b_1b_2}{4725}(xr_{0s})^{12}+\cdots\right]$$

$$E_2 = \frac{1}{2\pi\, G\rho_c}\frac{A_\psi}{\rho_\psi}\frac{\mathrm{d}P_\psi}{\mathrm{d}x}\frac{R\, x}{r_\psi^2}$$
$$= \frac{1}{2\pi\, G\, \rho_c\, R^2}\frac{1}{\rho_\psi}\left(\frac{1}{\rho_\psi}\frac{\mathrm{d}\rho_\psi}{\mathrm{d}x}-\frac{1}{\gamma\, P_\psi}\frac{\mathrm{d}P_\psi}{\mathrm{d}x}\right)\frac{\mathrm{d}P_\psi}{\mathrm{d}x}\frac{1}{x\, r_{0\,s}^3}\left[1-2b_0\,(xr_{0s})^3-\frac{16b_1}{15}(xr_{0s})^5\right.$$
$$-\frac{4b_0^2}{5}(xr_{0s})^6-\frac{16b_2}{21}(xr_{0s})^7-\frac{212b_0b_1}{105}(xr_{0s})^8-\left(\frac{704b_0b_2}{315}+\frac{1328b_1^2}{1575}\right)(xr_{0s})^{10}$$
$$\left.-\frac{1856b_1b_2}{1155}(xr_{0s})^{12}+\cdots\right]$$

$$E_3 = \frac{l}{x}+A_\psi\frac{\mathrm{d}r_\psi}{\mathrm{d}x} = \frac{l}{x}+\left(\frac{1}{\rho_\psi}\frac{\mathrm{d}\rho_\psi}{\mathrm{d}x}-\frac{1}{\gamma\, P_\psi}\frac{\mathrm{d}P_\psi}{\mathrm{d}x}\right), \qquad E_4 = \frac{l}{x}$$

$$F_1 = \frac{2l}{x}-\frac{\mathrm{d}^2r_\psi/\mathrm{d}x^2}{\mathrm{d}r_\psi/\mathrm{d}x}+\frac{2}{r_\psi}\frac{\mathrm{d}r_\psi}{\mathrm{d}x}$$
$$= \frac{1}{x}\left[2(l+1)-\left\{2b_0(xr_{0s})^3+\frac{8b_1}{3}(xr_{0s})^5+8b_0^2(xr_{0s})^6+\frac{16b_2}{5}(xr_{0s})^7\right.\right.$$
$$\left.\left.+\frac{96b_0b_1}{5}(xr_{0s})^8+\left(\frac{512b_0b_2}{21}+\frac{2864b_1^2}{315}\right)(xr_{0s})^{10}+\frac{31\,744b_1b_2}{1575}(xr_{0s})^{12}+\cdots\right\}\right]$$

$$F_2 = 2\frac{\rho_\psi}{\rho_c}\frac{A_\psi\, Rx}{r_\psi^2}\left(\frac{\mathrm{d}r_\psi}{\mathrm{d}x}\right)^2$$
$$= 2\frac{\rho_\psi}{\rho_c}\left(\frac{1}{\rho_\psi}\frac{\mathrm{d}P_\psi}{\mathrm{d}x}-\frac{1}{\gamma P_\psi}\frac{\mathrm{d}P_\psi}{\mathrm{d}x}\right)\frac{1}{x\, r_{0\,s}}\left[1+\frac{2b_0}{3}(xr_{0s})^3+\frac{8b_1}{15}(xr_{0s})^5+\frac{14b_0^2}{9}(xr_{0s})^6\right.$$
$$+\frac{16b_2}{35}(xr_{0s})^7+\frac{124b_0b_1}{45}(xr_{0s})^8+\left(\frac{96b_0b_2}{35}+\frac{184b_1^2}{175}\right)(xr_{0s})^{10}$$
$$\left.+\frac{9664b_1b_2}{4725}(xr_{0s})^{12}+\cdots\right]$$

$$F_3 = -\frac{4\pi\, G\rho_\psi^2}{\gamma\, P_\psi}\left(\frac{\mathrm{d}r_\psi}{\mathrm{d}x}\right)^2$$
$$= -\frac{4\pi\, G\, r_{0s}^2\, R^2\rho_\psi^2}{\gamma\, P_\psi}\left[1+\frac{8b_0}{3}(xr_{0s})^3+\frac{8b_1}{5}(xr_{0s})^5+\frac{346b_0^2}{45}(xr_{0s})^6+\frac{128b_2}{105}(xr_{0s})^7\right.$$
$$\left.+\frac{1136b_0b_1}{105}(xr_{0s})^8+\left(\frac{992b_0b_2}{105}+\frac{5672b_1^2}{1575}\right)(xr_{0s})^{10}+\left(\frac{314\,432b_1b_2}{51\,975}\right)(xr_{0s})^{12}+\cdots\right]$$

$$F_4 = \frac{l\,(l+1)}{x^2} - \frac{l}{x}\left(\frac{\mathrm{d}^2 r_\psi}{\mathrm{d}x^2}\right)\Big/\left(\frac{\mathrm{d}r_\psi}{\mathrm{d}x}\right) + \frac{2\,l}{x}\left(\frac{1}{r_\psi}\frac{\mathrm{d}r_\psi}{\mathrm{d}x}\right) - \frac{l\,(l+1)}{r_\psi^2}\left(\frac{\mathrm{d}r_\psi}{\mathrm{d}x}\right)^2$$

$$= -\frac{l}{x^2}\left\{\left[4b_0\,(xr_{0s})^3 + 4b_1\,(xr_{0s})^5 + \frac{67b_0^2}{5}(xr_{0s})^6 + \frac{64b_2}{15}\,(xr_{0s})^7 + \frac{964b_0b_1}{35}\,(xr_{0s})^8\right.\right.$$
$$+\left(\frac{10\,096b_0b_2}{315} + \frac{3796b_1^2}{315}\right)(xr_{0s})^{10} + \frac{133\,2416b_1b_2}{51\,975}(xr_{0s})^{12} + \cdots\bigg]\,l\,\big[2b_0\,(xr_{0s})^3$$
$$+\frac{4b_1}{3}\,(xr_{0s})^5 + \frac{27b_0^2}{5}(xr_{0s})^6 + \frac{16b_2}{15}(xr_{0s})^7 + \frac{292b_0b_1}{35}\,(xr_{0s})^8$$
$$\left.\left.+\left(\frac{2416b_0b_2}{315} + \frac{932b_1^2}{315}\right)(xr_{0s})^{10} + \frac{284\,864b_1b_2}{51\,975}(xr_{0s})^{12} + \cdots\right]\right\}.$$

Again σ is the eigenfrequency of oscillations and $x = r_0/r_{0s}$ is the nondimensional form of a distance of a fluid element from the center of the star. Also

$$\zeta = \frac{r_\psi^2\,\delta r_\psi}{R^3 x^{l+1}}, \qquad \eta = \frac{\mathrm{P}'_\psi}{2\pi\,\mathrm{G}\,\rho_\mathrm{c}\,R^2 x^l \rho_\psi} \qquad \text{and} \qquad \phi = \frac{\psi'_g}{2\,\pi\,\mathrm{G}\rho_\mathrm{c}\,R^2 x^l} \tag{A.5}$$

where δr_ψ being the amplitudes of Lagrangian variations in the radial direction and P'_ψ, ψ'_g the amplitudes of Lagrangian variation in pressure and gravitational potential on the equipotential surface ψ = constant.

The eigenvalue problem (A.4) determining the eigenfrequencies of nonradial modes of oscillations of a differentially rotating stellar model is to be solved subject to the boundary conditions at the center and the free surface. Boundary conditions at the center require $\delta r_\psi = 0$, $P'_\psi/\rho_\psi = 0$ and $\psi'_g = 0$ for $r_\psi = 0$. These requirements lead us to the analytic conditions

$$\eta + \phi = \frac{\sigma^2}{2\,\pi\,G\,\rho_c\,l\,r_{OS}}\zeta, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}x} = 0 \tag{A.6}$$

at the center $x = 0$.

If the pressure P_ψ on the free surface is taken to be zero, then δP_ψ, the Lagrangian variation in pressure, should be zero at the outer surface. This leads us to the condition

$$2\,\pi\,G\rho_c\,r_\psi^2\,\rho_\psi\,\frac{\mathrm{d}r_\psi}{\mathrm{d}x}\eta + R\frac{\mathrm{d}P_\psi}{\mathrm{d}x}\zeta = 0$$

or

$$2\,\pi\,G\,\rho_c\rho_\psi R^2 r_{0s}^3\left[1 + 4n\,r_{0s}^3 + \left(\frac{36}{5}q^2 + \frac{864}{45}n^2 + \frac{72}{15}nq\right)r_{0s}^6 + \frac{55q^2}{7}r_{0s}^8\right.$$
$$\left.+\frac{26q^2}{3}r_{0s}^{10} + \cdots\right]\eta + \frac{\mathrm{d}P_\psi}{\mathrm{d}x}\zeta = 0. \tag{A.7a}$$

However, the condition requiring the gravitational potential to be continuous across the free surface gives

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} + \left[l + \frac{(l+1)}{r_\psi}\frac{\mathrm{d}r_\psi}{\mathrm{d}x}\right]\phi + \frac{2\,R\,\rho_\psi}{\rho_c r_\psi^2}\frac{\mathrm{d}r_\psi}{\mathrm{d}x}\zeta = 0$$

or

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} + \phi\left\{l + (l+1)\left[1 + 2\,n\,r_{0s}^3 + \left(\frac{24q^2}{5} + \frac{396n^2}{45} + \frac{48nq}{15}\right)r_{0s}^6 + \frac{40q^2}{7}r_{0s}^8\right.\right.$$
$$\left.\left.+\frac{20q^2}{3}r_{0s}^{10} + \cdots\right]\right\} + \frac{2\rho_\psi}{\rho_c\,r_{0\,s}}\left[1 + \frac{4}{3}n\,r_{0\,s}^3 + \left(4\,q^2 + \frac{8}{3}n\,q + \frac{56}{9}\,n^2\right)r_{0\,s}^6\right.$$
$$\left.+5\,q^2 r_{0s}^8 + 6\,q^2 r_{0s}^{10} + \cdots\right]\zeta = 0 \tag{A.7b}$$

at the surface $x = 1$.

Thus, in terms of the nondimensional eigenfunctions ζ, η and ϕ, the problem of determining the eigenfrequencies of nonradial modes of oscillation of differentially rotating stellar model reduces to solving the system of differential equation (A.4) subject to the boundary conditions (A.6) at the center and the boundary conditions (A.7) at the free surface.

Appendix B. Explicit expressions for various coefficients of eigenvalue problems (3) and (4) in the case of differentially rotating polytropic models

In this section we have presented the explicit expressions for the various coefficients H_1, H_2 etc and B_1, B_2, E_1 etc used in computing the pseudo-radial and nonradial modes of oscillations of polytropic models of stars respectively (cf section 2 and section 3).

B.1. Radial oscillations (equation (3))

$$\begin{aligned}
H_1 = 1 &- \frac{8b_0}{3}r_0^3 - \left(\frac{8b_1}{5} + \frac{4b_3}{5}\right)r_0^5 - \frac{26b_0^2}{45}r_0^6 \\
&- \left(\frac{128b_2}{105} + \frac{16b_4}{15} + \frac{64b_5}{105}\right)r_0^7 - \left(\frac{16b_0b_1}{7} - \frac{4b_0b_3}{7}\right)r_0^8 \\
&- \left(\frac{928b_0b_2}{315} - \frac{68b_0b_4}{315} + \frac{16b_0b_5}{45} + \frac{328b_1^2}{315} + \frac{122b_3^2}{63} - \frac{40b_1b_3}{63}\right)r_0^{10} \\
&- \left(\frac{22\,336b_1b_2}{10\,395} - \frac{32b_1b_4}{45} - \frac{32b_1b_5}{693} - \frac{6304b_2b_3}{10\,395}\right. \\
&\left. + \frac{4b_3b_4}{5} + \frac{1376b_3b_5}{3465}\right)r_0^{12} + \cdots
\end{aligned}$$

$$\begin{aligned}
H_2 = \frac{1}{r_0}\Bigg[4 &- \frac{32b_0}{3}r_0^3 - \left(\frac{116b_1}{15} + \frac{58b_3}{15}\right)r_0^5 - \frac{266b_0^2}{45}r_0^6 - \left(\frac{736b_2}{105} + \frac{92b_4}{15} + \frac{368b_5}{105}\right)r_0^7 \\
&- \left(\frac{160b_0b_1}{9} - \frac{16b_0b_3}{9}\right)r_0^8 - \left(\frac{1504b_0b_2}{63} + \frac{52b_0b_4}{63} + \frac{176b_0b_5}{45}\right. \\
&\left. + \frac{904b_1^2}{105} + \frac{298b_3^2}{21} - \frac{24b_1b_3}{7}\right)r_0^{10} - \left(\frac{40\,064b_1b_2}{2079} - \frac{64b_1b_4}{15} + \frac{192b_1b_5}{385}\right. \\
&\left. - \frac{42\,304b_2b_3}{10\,395} + \frac{328b_3b_4}{45} + \frac{12\,736b_3b_5}{3465}\right)r_0^{12} + \cdots\Bigg]
\end{aligned}$$

$$H_3 = \frac{(N+1)\xi_u^2}{3\gamma r_{os}^3}\left(\frac{\overline{\rho}}{\rho_c}\right)\frac{1}{\theta_\psi}$$

$$\begin{aligned}
H_4 = -\left(3 - \frac{4}{\gamma}\right)\frac{(N+1)}{r_0}\left(\frac{1}{\theta_\psi}\frac{\mathrm{d}\theta_\psi}{\mathrm{d}r_0}\right)\Bigg[1 &- \frac{5b_0}{3}r_0^3 - \left(\frac{14b_1}{15} + \frac{7b_3}{15}\right)r_0^5 - \frac{47b_0^2}{45}r_0^6 \\
&- \left(\frac{24b_2}{35} + \frac{3b_4}{5} + \frac{12b_5}{35}\right)r_0^7 - \left(\frac{136b_0b_1}{63} + \frac{8b_0b_3}{63}\right)r_0^8 \\
&- \left(\frac{16b_0b_2}{7} + \frac{10b_0b_4}{21} + \frac{8b_0b_5}{15} + \frac{268b_1^2}{315} + \frac{71b_3^2}{63} - \frac{4b_1b_3}{63}\right)r_0^{10} \\
&- \left(\frac{2368b_1b_2}{1485} + \frac{32b_1b_5}{165} - \frac{32b_2b_3}{297} + \frac{28b_3b_4}{45} + \frac{32b_3b_5}{99}\right)r_0^{12} + \cdots\Bigg]
\end{aligned}$$

where ξ_u is the value of ξ at the outer surface of the undistorted polytropic model (where ξ is the Lane–Emden variable, specifically we have the value of $\xi_u = 3.653\,75,\ 6.896\,85$

corresponding to polytropic index $N = 1.5$ and 3.0 respectively), ρ_c the density at the center and $\overline{\rho}$ the average density of the undistorted polytropic model, N is the polytropic index of the model, γ is the ratio of specific heats, θ_ψ is the parameter depending upon the distance of the chosen point from the center of the star.

Assuming that the angular velocity of differential rotation of the star is not too large, in the above expressions, terms up to second order of smallness in $b_0, b_1, b_2, b_3, b_4, b_5$ and up to order r_0^{12} in r_0 have been only retained.

On setting $b_3 = 0, b_4 = 0, b_5 = 0$ in the above expressions, we get same expressions as obtained earlier by Mohan *et al* (1998) (equation (7) of paper II) in which the differential rotation was assumed to depend on the parameter s alone.

B.2. Nonradial oscillations (equation (4))

$$B_1 = \frac{l+1}{x} + \frac{N+1}{\gamma}\left(\frac{1}{\theta_\psi}\frac{\mathrm{d}\theta_\psi}{\mathrm{d}x}\right)$$

$$\begin{aligned}
B_2 = \frac{(N+1)\xi_u^2\, x\, r_{0s}^3}{2\,\gamma\,\theta_\psi}\Bigg[&1 + 2b_0(xr_{0s})^3 + \left(\frac{16b_1}{15} + \frac{8b_3}{15}\right)(xr_{0s})^5 + \frac{24b_0^2}{5}(xr_{0s})^6 \\
&+ \left(\frac{16b_2}{21} + \frac{2b_4}{3} + \frac{8b_5}{21}\right)(xr_{0s})^7 + \left(\frac{44b_0b_1}{7} + \frac{44b_0b_3}{21}\right)(xr_{0s})^8 \\
&+ \left(\frac{1664b_0b_2}{315} + \frac{104b_0b_4}{35} + \frac{208b_0b_5}{105} + \frac{104b_1b_3}{105} + \frac{208b_1^2}{105} + \frac{52b_3^2}{35}\right)(xr_{0s})^{10} \\
&+ \left(\frac{2240b_1b_2}{693} + \frac{203b_1b_4}{150} + \frac{224b_1b_5}{231} + \frac{448b_2b_3}{693}\right. \\
&\left. + \frac{4b_3b_4}{3} + \frac{22\,072b_3b_5}{3465}\right)(xr_{0s})^{12} + \cdots\Bigg]
\end{aligned}$$

$$\begin{aligned}
B_3 = -\frac{3\,l\,(l+1)r_{0s}^4}{2x}\left(\frac{\rho_c}{\overline{\rho}}\right)\Bigg[&1 + \frac{4b_0}{3}(xr_{0s})^3 + \left(\frac{4b_1}{5} + \frac{2b_3}{5}\right)(xr_{0s})^5 + \frac{133b_0^2}{45}(xr_{0s})^6 \\
&+ \left(\frac{64b_2}{105} + \frac{8b_4}{15} + \frac{32b_5}{105}\right)(xr_{0s})^7 + \left(\frac{152b_0b_1}{35} + \frac{46b_0b_3}{35}\right)(xr_{0s})^8 \\
&+ \left(\frac{1232b_0b_2}{315} + \frac{638b_0b_4}{315} + \frac{88b_0b_5}{63} + \frac{1012b_1b_3}{1575} + \frac{2332b_1^2}{1575} + \frac{1903b_3^2}{1575}\right)(xr_{0s})^{10} \\
&+ \left(\frac{131\,872b_1b_2}{51\,975} + \frac{208b_1b_4}{225} + \frac{12\,272b_1b_5}{17\,325} + \frac{22\,256b_2b_3}{51\,975}\right. \\
&\left. + \frac{26b_3b_4}{25} + \frac{9776b_3b_5}{17\,325}\right)(xr_{0s})^{12} + \cdots\Bigg]
\end{aligned}$$

$$\begin{aligned}
E_1 = -\frac{2}{3xr_{0s}^4}\left(\frac{\overline{\rho}}{\rho_c}\right)\Bigg[&1 + \frac{2b_0}{3}(xr_{0s})^3 + \left(\frac{8b_1}{15} + \frac{4b_3}{15}\right)(xr_{0s})^5 + \frac{14b_0^2}{9}(xr_{0s})^6 \\
&+ \left(\frac{16b_2}{35} + \frac{2b_4}{5} + \frac{8b_5}{35}\right)(xr_{0s})^7 + \left(\frac{124b_0b_1}{45} + \frac{32b_0b_3}{45}\right)(xr_{0s})^8
\end{aligned}$$

$$+\left(\frac{96b_0b_2}{35}+\frac{44b_0b_4}{35}+\frac{32b_0b_5}{35}+\frac{64b_1b_3}{175}+\frac{184b_1^2}{175}+\frac{166b_3^2}{175}\right)(xr_{0s})^{10}$$
$$+\left(\frac{9664b_1b_2}{4725}+\frac{52b_1b_4}{75}+\frac{96b_1b_5}{175}+\frac{1472b_2b_3}{4725}\right.$$
$$\left.+\frac{188b_3b_4}{225}+\frac{712b_3b_5}{1575}\right)(xr_{0s})^{12}+\cdots\Bigg]$$

$$E_2=\frac{2}{\xi_u^2}\left(N-\frac{N+1}{\gamma}\right)\frac{1}{\theta_\psi}\left(\frac{\mathrm{d}\theta_\psi}{\mathrm{d}x}\right)^2\frac{1}{xr_{0s}^3}\left[1-2b_0\,(xr_{0s})^3-\left(\frac{16b_1}{15}+\frac{8b_3}{15}\right)(xr_{0s})^5\right.$$
$$-\frac{4b_0^2}{5}(xr_{0s})^6-\left(\frac{16b_2}{21}+\frac{2b_4}{3}+\frac{8b_5}{21}\right)(xr_{0s})^7-\left(\frac{212b_0b_1}{105}-\frac{4b_0b_3}{105}\right)(xr_{0s})^8$$
$$-\left(\frac{704b_0b_2}{315}+\frac{32b_0b_4}{105}+\frac{16b_0b_5}{35}-\frac{232b_1b_3}{1575}+\frac{1328b_1^2}{1575}+\frac{1892b_3^2}{1575}\right)(xr_{0s})^{10}$$
$$-\left(\frac{1856b_1b_2}{1155}-\frac{31b_1b_4}{450}+\frac{544b_1b_5}{3465}-\frac{64b_2b_3}{385}+\frac{28b_3b_4}{45}\right.$$
$$\left.\left.+\frac{328b_3b_5}{55}\right)(xr_{0s})^{12}+\cdots\right]$$

$$E_3=\frac{l}{x}+\left(N-\frac{N+1}{\gamma}\right)\frac{1}{\theta_\psi}\left(\frac{\mathrm{d}\theta_\psi}{\mathrm{d}x}\right),\qquad E_4=\frac{l}{x}$$

$$F_1=\frac{1}{x}\left[2(l+1)-\left\{2b_0(xr_{0s})^3+\left(\frac{8b_1}{3}+\frac{4b_3}{3}\right)(xr_{0s})^5+8b_0^2(xr_{0s})^6\right.\right.$$
$$+\left(\frac{16b_2}{5}+\frac{14b_4}{5}+\frac{8b_5}{5}\right)(xr_{0s})^7+\left(\frac{96b_0b_1}{5}+\frac{64b_0b_3}{15}\right)(xr_{0s})^8$$
$$+\left(\frac{512b_0b_2}{21}+\frac{208b_0b_4}{21}+\frac{160b_0b_5}{7}+\frac{704b_1b_3}{315}+\frac{2864b_1^2}{315}+\frac{2876b_3^2}{315}\right)(xr_{0s})^{10}$$
$$+\left(\frac{31744b_1b_2}{1575}+\frac{112b_1b_4}{25}+\frac{768b_1b_5}{175}+\frac{544b_2b_3}{315}+\frac{608b_3b_4}{75}\right.$$
$$\left.\left.\left.+\frac{2272b_3b_5}{525}\right)(xr_{0s})^{12}+\cdots\right\}\right]$$

$$F_2=\frac{2}{xr_{0s}}\left(N-\frac{N+1}{\gamma}\right)\theta_\psi^{N-1}\frac{\mathrm{d}\theta_\psi}{\mathrm{d}x}\left[1+\frac{2b_0}{3}(xr_{0s})^3+\left(\frac{8b_1}{15}+\frac{4b_3}{15}\right)(xr_{0s})^5+\frac{14b_0^2}{9}(xr_{0s})^6\right.$$
$$+\left(\frac{16b_2}{35}+\frac{2b_4}{5}+\frac{8b_5}{35}\right)(xr_{0s})^7+\left(\frac{124b_0b_1}{45}+\frac{32b_0b_3}{45}\right)(xr_{0s})^8$$
$$+\left(\frac{96b_0b_2}{35}+\frac{44b_0b_4}{35}+\frac{32b_0b_5}{35}+\frac{64b_1b_3}{175}+\frac{184b_1^2}{175}+\frac{166b_3^2}{175}\right)(xr_{0s})^{10}$$
$$+\left(\frac{9664b_1b_2}{4725}+\frac{52b_1b_4}{75}+\frac{96b_1b_5}{175}+\frac{1472b_2b_3}{4725}\right.$$
$$\left.\left.+\frac{188b_3b_4}{225}+\frac{712b_3b_5}{1575}\right)(xr_{0s})^{12}+\cdots\right]$$

$$
\begin{aligned}
F_3 = -\frac{(N+1)}{\gamma}\,\xi_u^2\theta_\psi^{N-1}r_{0s}^2\Bigg[&1+\frac{8b_0}{3}(xr_{0s})^3+\left(\frac{8b_1}{5}+\frac{4b_3}{5}\right)(xr_{0s})^5+\frac{346b_0^2}{45}(xr_{0s})^6\\
&+\left(\frac{128b_2}{105}+\frac{16b_4}{15}+\frac{64b_5}{105}\right)(xr_{0s})^7+\left(\frac{1136b_0b_1}{105}+\frac{388b_0b_3}{105}\right)(xr_{0s})^8\\
&+\left(\frac{992b_0b_2}{105}+\frac{1724b_0b_4}{315}+\frac{1136b_0b_5}{315}+\frac{3032b_1b_3}{1575}+\frac{5672b_1^2}{1575}+\frac{4058b_3^2}{1575}\right)(xr_{0s})^{10}\\
&+\left(\frac{314\,432b_1b_2}{51\,975}+\frac{608b_1b_4}{225}+\frac{32\,992b_1b_5}{17\,325}+\frac{69\,856b_2b_3}{51\,975}\right.\\
&\left.+\frac{188b_3b_4}{75}+\frac{23\,776b_3b_5}{17\,325}\right)(xr_{0s})^{12}+\cdots\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
F_4 = -\frac{l}{x^2}\Bigg\{\Bigg[&4b_0(xr_{0s})^3+(4b_1+2b_3)(xr_{0s})^5+\frac{67b_0^2}{5}(xr_{0s})^6+\left(\frac{64b_2}{15}+\frac{56b_4}{15}+\frac{32b_5}{15}\right)(xr_{0s})^7\\
&+\left(\frac{964b_0b_1}{35}+\frac{242b_0b_3}{35}\right)(xr_{0s})^8+\left(\frac{10\,096b_0b_2}{315}+\frac{1478b_0b_4}{105}+\frac{3287b_0b_5}{315}\right.\\
&\left.+\frac{1156b_1b_3}{315}+\frac{3796b_1^2}{315}+\frac{3589b_3^2}{315}\right)(xr_{0s})^{10}+\left(\frac{133\,2416b_1b_2}{51975}+\frac{1504b_1b_4}{225}\right.\\
&\left.+\frac{104\,096b_1b_5}{17325}+\frac{144\,832b_2b_3}{51975}\,\frac{2332b_3b_4}{225}+\frac{32096b_3b_5}{5775}\right)(xr_{0s})^{12}+\cdots\Bigg]\\
&+l\Bigg[2b_0(xr_{0s})^3+\left(\frac{4b_1}{3}+\frac{2b_3}{3}\right)(xr_{0s})^5+\frac{27b_0^2}{5}(xr_{0s})^6\\
&+\left(\frac{16b_2}{15}+\frac{14b_4}{15}+\frac{8b_5}{15}\right)(xr_{0s})^7+\left(\frac{292b_0b_1}{35}+\frac{278b_0b_3}{105}\right)(xr_{0s})^8\\
&+\left(\frac{2416b_0b_2}{315}+\frac{146b_0b_4}{35}+\frac{887b_0b_5}{315}+\frac{452b_1b_3}{315}+\frac{932b_1^2}{315}+\frac{713b_3^2}{315}\right)(xr_{0s})^{10}\\
&+\left(\frac{284\,864b_1b_2}{51\,975}+\frac{496b_1b_4}{225}+\frac{28\,064b_1b_5}{17\,325}+\frac{55\,072b_2b_3}{51\,975}\right.\\
&\left.+\frac{508b_3b_4}{225}+\frac{2368b_3b_5}{1925}\right)(xr_{0s})^{12}+\cdots\Bigg]\Bigg\}.
\end{aligned}
$$

Also

$$
\omega^{\bullet 2}=\frac{R^3r_{os}^3\sigma^2}{GM_0},\qquad x=\frac{r_0}{r_{0s}},\qquad \zeta=\frac{r_\psi^2\,\delta r_\psi}{R^3x^{l+1}},
$$

$$
\eta=\frac{P'_\psi}{2\pi G\rho_cR^2x^lP_\psi}\qquad\text{and}\qquad \phi=\frac{\psi'_g}{2\pi G\rho_cR^2x^l}
$$

where δr_ψ being the amplitude of Lagrangian variation in r_ψ, P'_ψ the amplitude of variation of P_ψ and ψ'_g the amplitude of variation of the gravitational potential ψ_g at a point on the topologically equivalent spherical equipotential surface ψ = constant. Other symbols have the same meaning as assigned earlier in the paper.

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